Some classes of rational functions and related Banach spaces

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Throughout this paper, d and r will be positive integers and $0 < \delta < 1$. These numbers will be arbitrary with the given properties unless they are further specified.

For a matrix or vector X let X' denote its transpose. Thus if $x = (x_1, ..., x_d)' \in \mathbb{R}^d$ is a column vector and A is a $d \times d$ matrix, then $x'Ax$ gives the quadratic form defined by the matrix A, $\sum_{i,j=1}^{n} A_{ij}x_ix_j$. Let $\mathcal{P}_d$ denote the set of symmetric $d \times d$ positive definite matrices. Let $\|A\|$ be the usual operator norm of a matrix A, $\|A\| := \sup\{|Ax| : |x| = 1\}$, where $|\cdot|$ is the usual Euclidean norm on $\mathbb{R}^d$. Recall that $\mathbb{N}$ is the set of nonnegative integers. Let $\mathcal{MM}_r := \mathcal{MM}_{r,d}$ be the set of monic monomials from $\mathbb{R}^d$ into $\mathbb{R}$ of degree r, namely the set of all functions $g(x) = \Pi_{i=1}^{d} x_i^{n_i}$ with $n_i \in \mathbb{N}$ and $\sum_{i=1}^{d} n_i = r$. Let $\mathcal{W}_\delta := \mathcal{W}_{\delta,d} := \{C \in \mathcal{P}_d : \|C\| < 1/\delta, \|C^{-1}\| < 1/\delta\}$. Let

$$\mathcal{F}_{\delta,r} := \mathcal{F}_{\delta,r,d} := \left\{ f : \mathbb{R}^d \to \mathbb{R}, \ f(x) \equiv g(x)/\Pi_{s=1}^{r}(1 + x'C_sx), \right.$$

$$\left. \text{where } g \in \mathcal{MM}_{2r}, \text{ and for } s = 1, ..., r, \ C_s \in \mathcal{W}_\delta \right\}.$$

Let $\mathcal{G}_{\delta,r} := \bigcup_{v=1}^{r} \mathcal{F}_{\delta,v}$. For $1 \leq j \leq r$, let $\mathcal{F}_{\delta,r}^{(j)}$ be the set of $f \in \mathcal{F}_{\delta,r}$ such that C_s has at most j different values. We will be interested in $j = 1$ and 2. Clearly $\mathcal{F}_{\delta,r}^{(1)} \subset \mathcal{F}_{\delta,r}^{(2)} \subset \cdots \subset \mathcal{F}_{\delta,r}$ for each δ and r.

Let $h_C(x) := 1 + x'Cx$ for $C \in \mathcal{P}_d$ and $x \in \mathbb{R}^d$. Then clearly $f \in \mathcal{F}_{\delta,r}^{(1)}$ if and only if for some $P \in \mathcal{MM}_{2r}$ and $C \in \mathcal{W}_\delta$ we have for all x

$$(1) \qquad f(x) = f_{P,C,r}(x) := P(x)h_C(x)^{-r}.$$

Each $C \in \mathcal{P}_d$ can be diagonalized for an orthonormal basis of eigenvectors with positive eigenvalues. For $C = C_s \in \mathcal{W}_\delta$ as in the definition of $\mathcal{F}_{\delta,r}$ we have for all $x \in \mathbb{R}^d$ that $\delta|x|^2 \leq x'Cx \leq |x|^2/\delta$. Set $\|f\|_{\sup} := \sup\{|f(x)| : x \in \mathbb{R}^d\}$. Recall that $u \vee w := \max(u, w)$ and $u \wedge w := \min(u, w)$.

Lemma 1. *For any $d = 1, 2, ...,$ $r = 1, 2, ...,$ $0 < \delta < 1$, and $f \in \mathcal{G}_{\delta,r}$ we have*
(a) $\|f\|_{\sup} \leq \delta^{-r}$,
(b) $\|f\|_{\sup} \geq (\delta/d)^r$.

Proof. First let $f \in \mathcal{F}_{\delta,r}$. For (a), we have for all x that

$$|f(x)| \leq (1 \vee |x|)^{2r}/(1 + \delta|x|^2)^r = (1 \vee |x|^2)^r/(1 + \delta|x|^2)^r \leq \delta^{-r},$$

by considering the two cases $|x| \geq 1$ and $|x| < 1$. So (a) follows in this case.

For (b), for any M with $1 \leq M < \infty$ consider the point x where $x_1 = x_2 = \cdots = x_d = M$. For this x we get

$$\|f\|_{\sup} \geq |f(x)| \geq M^{2r}/(1 + (dM^2/\delta))^r = [\delta M^2/(\delta + dM^2)]^r \to (\delta/d)^r$$

as $M \to +\infty$, proving (b) for $f \in \mathcal{F}_{\delta,r}$.

Now for any $f \in \mathcal{G}_{\delta,r}$, we have $f \in \mathcal{F}_{\delta,v}$ for some $v = 1, ..., r$, $\delta^{-v} \leq \delta^{-r}$ and $(\delta/d)^v \geq (\delta/d)^r$, so the lemma follows. $\qquad\square$

Lemma 2. *For any $d = 1, 2, ...,$ $r = 1, 2, ...,$ and $0 < \delta < 1$, let $f = f_{P,C,r}$ and $g = f_{P,D,r}$ for some $P \in \mathcal{MM}_{2r}$ and $C, D \in \mathcal{P}_d$. Then*

$$(2) \qquad (f - g)(x) \equiv \frac{x'(D - C)xP(x)\sum_{j=0}^{r-1} h_D(x)^{r-1-j}h_C(x)^j}{(h_Ch_D)(x)^r}.$$

For $1 \leq k \leq l \leq d$ and $j = 0, 1, ..., r - 1$, let

$$
\begin{aligned}
h_{C,D,k,l,r,j}(x) \quad &:= \quad x_k x_l P(x) h_D(x)^{r-1-j} h_C(x)^j / (h_C h_D)(x)^r \\
&= \quad x_k x_l P(x) h_C(x)^{j-r} h_D(x)^{-j-1}.
\end{aligned}
$$

Then each $h_{C,D,k,l,r,j}$ is in $\mathcal{F}_{\delta,r+1}^{(2)}$ and

$$(3) \qquad g - f \equiv -\sum_{1 \leq k \leq l \leq d}\sum_{j=0}^{r-1}(D_{kl} - C_{kl})(2 - \delta_{kl})h_{C,D,k,l,r,j}.$$

Proof. Since $P(x)$ is a common factor we need only note that

$$h_C(x)^{-r} - h_D(x)^{-r} = (h_D(x)^r - h_C(x)^r)/(h_Ch_D)(x)^r$$

and expand the numerator by the identity

$$U^r - V^r = (U - V)\sum_{j=0}^{r-1}U^{r-1-j}V^j$$

for any two real numbers U and V to get (2).

The functions $h_{C,D,k,l,r,j}$ are clearly in $\mathcal{F}_{\delta,r+1}^{(2)}$, and (3) follows straightforwardly. $\qquad\square$

For $j = 1, 2, ...,$ let $\mathcal{G}_{\delta,r}^{(j)} := \bigcup_{v=1}^{r}\mathcal{F}_{\delta,v}^{(j)}$. For any $f : \mathbb{R}^d \to \mathbb{R}$, define

$$\|f\|_{\delta,r}^{*,j} := \|f\|_{\delta,r,d}^{*,j}$$

$$:= \inf\left\{\sum_{s=1}^{\infty} |\lambda_s| : \exists g_s \in \mathcal{G}_{\delta,r}^{(j)}, \ s \geq 1, \ f \equiv \sum_{s=1}^{\infty} \lambda_s g_s\right\},$$

or $+\infty$ if no such λ_s, g_s with $\sum_s |\lambda_s| < \infty$ exist. Lemma 1(a) implies that for $\sum_s |\lambda_s| < \infty$ and $g_s \in \mathcal{G}_{\delta,r} \equiv \mathcal{G}_{\delta,r}^{(r)}$, $\sum_s \lambda_s g_s$ converges absolutely and uniformly on $\mathbb{R}^d$. Let $Y_{\delta,r}^j := Y_{\delta,r,d}^j$ be the set of all functions f from $\mathbb{R}^d$ into $\mathbb{R}$ such that $\|f\|_{\delta,r}^{*,j} < \infty$. It's easily seen that each $Y_{\delta,r}^j$ is a real vector space of functions on $\mathbb{R}^d$ and $\|\cdot\|_{\delta,r}^{*,j}$ is a seminorm on it.

Lemma 3. *For any $j = 1, 2, ...,$*
(a) If $f \in \mathcal{G}_{\delta,r}^{(j)}$ then $f \in Y_{\delta,r}^j$ and $\|f\|_{\delta,r}^{*,j} \leq 1$.
(b) For any $g \in Y_{\delta,r}^j$, $\|g\|_{\sup} \leq \|g\|_{\delta,r}^{*,j}/\delta^r < \infty$.
(c) If $f \in \mathcal{G}_{\delta,r}^{(j)}$ then $\|f\|_{\delta,r}^{*,j} \geq (\delta^2/d)^r$.
(d) $\|\cdot\|_{\delta,r}^{*,j}$ is a norm on $Y_{\delta,r}^j$.
(e) $Y_{\delta,r}^j$ is complete for $\|\cdot\|_{\delta,r}^{*,j}$ and thus a Banach space.

Proof. Part (a) is clear. For part (b), let $g = \sum_s \lambda_s g_s$ with $g_s \in \mathcal{G}_{\delta,r}^{(j)}$ and apply Lemma 1(a) to each g_s. Part (c) follows from part (b) and Lemma 1(b). For part (d) we already noted that $\|\cdot\|_{\delta,r}^{*,j}$ is a seminorm. By part (b), if $\|g\|_{\delta,r}^{*,j} = 0$ then $\|g\|_{\sup} = 0$ so $g \equiv 0$, hence $\|\cdot\|_{\delta,r}^{*,j}$ is a norm.

For part (e), let $\{f_k\}_{k\geq 1}$ be a Cauchy sequence in $Y_{\delta,r}^j$ for $\|\cdot\|_{\delta,r}^{*,j}$. Then by part (b) it is also a Cauchy sequence for $\|\cdot\|_{\sup}$ and so converges uniformly on $\mathbb{R}^d$ to some function f. Taking a subsequence, we get f_{k_i} such that $\|f_{k_i} - f_k\|_{\delta,r}^{*,j} < 1/2^i$ for all $k \geq k_i$ for $i = 1, 2, ...$ Then the series $f_{k_1} + \sum_{i=1}^{\infty} f_{k_{i+1}} - f_{k_i}$ converges in sup norm and in $\|\cdot\|_{\delta,r}^{*,j}$ to f, writing $f_{k_{i+1}} - f_{k_i} = \sum_s \lambda_{is} f_{is}$ for some $f_{is} \in \mathcal{G}_{\delta,r}^{(j)}$ and $\sum_s |\lambda_{is}| \leq 1/2^i$, so that $\sum_{i,s} |\lambda_{is}| \leq 1$. It follows that $f \in Y_{\delta,r}^j$. This finishes the proof. $\square$

Lemma 4. *For any $j = 1, 2, ...,$ we have $Y_{\delta,r}^j \subset Y_{\delta,r+1}^j$. The inclusion linear map from $Y_{\delta,r}^j$ into $Y_{\delta,r+1}^j$ has norm at most 1.*

Proof. Simply $\mathcal{G}_{\delta,r}^{(j)} \subset \mathcal{G}_{\delta,r+1}^{(j)}$ and so $Y_{\delta,r}^j \subset Y_{\delta,r+1}^j$ with the inclusion map having norm bounded above by 1. $\square$

Remark. Let spaces $Z_{\delta,r}^j$ be defined like spaces $Y_{\delta,r}^j$ but with $\mathcal{F}_{\delta,r}^{(j)}$ in place of $\mathcal{G}_{\delta,r}^{(j)}$. Then the inclusion $Z_{\delta,r}^j \subset Z_{\delta,r+1}^j$ doesn't hold, for example $x^2/(1+x^2)$ is in $Z_{\delta,1}^1$ but not in $Z_{\delta,2}^1$: suppose $x^2/(1+x^2) = \sum_s \lambda_s x^4/(1+a_s x^2)^2$ with $\sum_s |\lambda_s| < \infty$ and $\delta < a_s < 1/\delta$ for all s.

Dividing both sides by x^2 and letting $x \to 0$, the left side approaches 1 and the right side 0.

In partial derivatives $\partial/\partial C_{kl}$ with respect to elements of a symmetric matrix C, $C_{lk} \equiv C_{kl}$ will vary while all other elements C_{vw} are held fixed.

Proposition 1. *Let $P \in \mathcal{MM}_{2r}$, and consider the function $\phi(C, x) := f_{P,C,r}(x) = P(x)/h_C(x)^r$ from $\mathcal{W}_\delta \times \mathbb{R}^d$ into $\mathbb{R}$. Then:*
(a) For each fixed $C \in \mathcal{W}_\delta$, $\phi(C, \cdot) \in \mathcal{F}_{\delta,r}^{(1)}$.
(b) With respect to any entry C_{kl} of C, $\phi(\cdot, x)$ has the partial derivative

$$\frac{\partial \phi(C, x)}{\partial C_{kl}} = -\frac{r(2 - \delta_{kl})x_k x_l P(x)}{h_C(x)^{r+1}}.$$

(c) The map $C \mapsto \partial\phi(C, \cdot)/\partial C_{kl}$ is Lipschitz from $\mathcal{W}_\delta$ into $Y_{\delta,r+2}^2$ for each $k, l = 1, ..., d$.
(d) The map $C \mapsto \phi(C, \cdot)$ from $\mathcal{W}_\delta$ into $\mathcal{F}_{\delta,r}^{(1)} \subset Y_{\delta,r}^1$, viewed as a map into the larger space $Y_{\delta,r+2}^2$, is Fréchet C^1.

Proof. Part (a) is clear, by (1). Part (b) follows by elementary calculus.

For part (c), $Y_{\delta,r}^1 \subset Y_{\delta,r}^2 \subset Y_{\delta,r+2}^2$ where the first inclusion is immediate and the second follows from Lemma 4 applied twice. From part (b), the given map takes values in $Y_{\delta,r+1}^1 \subset Y_{\delta,r+2}^1$ by Lemma 4. It is Lipschitz into $Y_{\delta,r+2}^2$ by Lemma 2 applied to $r + 1$ in place of r.

For part (d), and any $h \neq 0$ in $\mathbb{R}$, let $C_{k,l,h} \in \mathcal{W}_\delta$ for h small enough be the matrix which equals C except that h is added to C_{kl}, and also to C_{lk} if $k \neq l$. Applying Lemma 2, we see that $[\phi(C_{k,l,h}, \cdot) - \phi(C, \cdot)]/h \in Y_{\delta,r+1}^2$ for h small enough and equals, for $D = C_{k,l,h}$, at any $x \in \mathbb{R}^d$,

$$-\frac{(2 - \delta_{kl})x_k x_l P(x) \sum_{j=0}^{r-1} h_D(x)^{r-1-j} h_C(x)^j}{(h_C h_D)(x)^r}.$$

It will be shown that as D approaches C, i.e. $h \to 0$, the function in the last display converges in $Y_{\delta,r+2}^2$ to the corresponding function with D replaced by C. Considering one term at a time in the sum of r terms, we get

$$h_D(x)^{-j-1} h_C(x)^{j-r} - h_C(x)^{-r-1} = h_C(x)^{j-r}[h_D(x)^{-j-1} - h_C(x)^{-j-1}].$$

By the proof of Lemma 2,

$$h_D(x)^{-j-1} - h_C(x)^{-j-1} \equiv -h(2 - \delta_{kl})x_k x_l \sum_{i=0}^{j} h_C(x)^{i-j-1} h_D(x)^{-i-1}.$$

Thus we get

$$h(2 - \delta_{kl})^2 x_k^2 x_l^2 P(x) h_C(x)^{j-r} \sum_{i=0}^{j} h_C(x)^{i-j-1} h_D(x)^{-i-1} \ \to \ 0$$

in $Y_{\delta,r+2}^2$ as $h \to 0$. This implies existence of the partial derivatives

$$\partial/\partial C_{kl} \left[C \mapsto \phi(C, \cdot) \in Y_{\delta,r+2}^2 \right].$$

Part (c) gives their continuity. Continuous first partial derivatives imply that the function $C \mapsto \phi(C, \cdot)$ is Fréchet C^1 by known facts in analysis, completing the proof. $\qquad\square$

Theorem 1. *Let $r = 1, 2, ..., d = 1, 2, ..., 0 < \delta < 1$, and $f \in Y_{\delta,r}^1$, so that for some a_s with $\sum_s |a_s| < \infty$ we have $f(x) \equiv \sum_s a_s P_s(x)/(1 + x'C_s x)^{k_s}$ for $x \in \mathbb{R}^d$ where each $P_s \in \mathcal{MM}_{2k_s}$, $k_s = 1, ..., r$, and $C_s \in \mathcal{W}_\delta$. Then f can be written as a sum of the same form in which the triples (P_s, C_s, k_s) are all distinct. In that case, the C_s, P_s, k_s and the coefficients a_s are uniquely determined by f.*

Proof. Because $\sum |a_s| < \infty$, we can directly sum terms having the same P_s, C_s and k_s into one. To prove the second conclusion is equivalent to showing that then, if $f \equiv 0$ on $\mathbb{R}^d$ we have all $a_s = 0$.

Suppose the dimension $d = 1$. Then each C_s is a real number with $\delta < C_s < 1/\delta$, each $P_s(x) = x^{2k_s}$, and for each s the denominator is $(1 + C_s x^2)^{k_s}$. Each term in the sum is a rational function of x which we take as a rational function of a complex variable z. Via a partial fraction decomposition (e.g. Knopp, 1947, Chapter 2), we can rewrite the sum of at most r terms having a fixed value of $C_s = C$ as a sum

$$\gamma_C + \sum_{k=1}^{r} \frac{\alpha_{C,k}}{(z - z_C)^k} + \frac{\beta_{C,k}}{(z + z_C)^k}$$

where $z_C = i/\sqrt{C}$ and $\alpha_{C,k}$, $\beta_{C,k}$ are complex constants, γ_C real. It will suffice to prove that

(4) $$\sum_s \left[|\gamma_{C_s}| + \sum_{k=1}^{r} |\alpha_{C_s,k}| + |\beta_{C_s,k}| \right] \ < \ \infty,$$

and given that, to show that for $C = C_s$ and $k = k_s$ for any s,

(5) $$\alpha_{C,k} = \beta_{C,k} = 0,$$

because if we consider the largest value of $k = k_s$ with $a_s \neq 0$ and $C_s = C$, we will get $\alpha_{C,k} \neq 0$.

Proof of (4): We will show that when $\delta < C < 1/\delta$ and we take the following partial fraction decomposition,

$$(6) \qquad \frac{z^{2k}}{(1 + Cz^2)^k} = \frac{1}{C^k} + \sum_{v=1}^{k} \left(\frac{A_v}{(z - z_C)^v} + \frac{B_v}{(z + z_C)^v} \right),$$

then $\sum_v (|A_v| + |B_v|)$ has a suitable upper bound. In fact it will be shown in (11) to be bounded above by $2^k \delta^{-3k/2}$, and this will suffice.

Note that we have the term C^{-k} in the partial fraction decomposition (6) since this is the limit of the left-hand side as $z \to \infty$. Let $a_{C,k}$ be the coefficient of $x^{2k}/h_C(x)^k$ in the statement of Theorem 1. Then

$$|\gamma_C| \leq \sum_{k=1}^{r} C^{-k} |a_{C,k}| \leq \delta^{-k} \sum_{k=1}^{r} |a_{C,k}|$$

and $\sum_s |\gamma_{C_s}| < \infty$.

Using that $z_C = i/\sqrt{C}$, we multiply both sides of (6) by $(-C)^k$, then use $C = -z_C^{-2}$ to get

$$\frac{(z/z_C)^{2k}}{\left(1 - (z/z_C)^2\right)^k} = (-1)^k + \sum_{v=1}^{k} \left(\frac{A_v}{z_C^{2k+v} \left(\frac{z}{z_C} - 1\right)^v} + \frac{B_v}{z_C^{2k+v} \left(\frac{z}{z_C} + 1\right)^v} \right).$$

Let now $A'_v = A_v/z_C^{2k+v}$ and $B'_v = B_v/z_C^{2k+v}$. Note that A'_v and B'_v are the coefficients in the following decomposition:

$$\frac{\xi^{2k}}{(1 - \xi^2)^k} = (-1)^k + \sum_{v=1}^{k} \left(\frac{A'_v}{(\xi - 1)^v} + \frac{B'_v}{(\xi + 1)^v} \right).$$

To bound sums of absolute values of A'_v and B'_v, the following lemma will be proved:

Lemma 5. *(a) For any positive integers m and n, and any complex $z \neq \pm 1$, we have*

$$(7) \qquad \frac{1}{(1 - z)^m (1 + z)^n} = \sum_{j=1}^{m} \frac{a_j}{(1 - z)^j} + \sum_{i=1}^{n} \frac{b_i}{(1 + z)^i}$$

where $a_j := a_j^{m,n} > 0$ and $b_i := b_i^{m,n} > 0$ for all $j = 1, ..., m$ and $i = 1, ..., n$, and $\sum_{j=1}^{m} a_j + \sum_{i=1}^{n} b_i = 1$.
(b) We have for $k = 0, 1, ..., m - 1$

$$(8) \qquad a_{m-k} = \frac{1}{2^{n+k}} \binom{n + k - 1}{k}.$$

(c) For $i = 0, 1, \ldots, n - 1$

(9)
$$b_{n-i} = \frac{1}{2^{m+i}} \binom{m+i-1}{i}.$$

(d) For each positive integer k and $z \neq \pm 1$,

(10)
$$\frac{z^{2k}}{(1-z^2)^k} = (-1)^k + \sum_{j=1}^{k} \alpha_j \left[\frac{1}{(1-z)^j} + \frac{1}{(1+z)^j} \right]$$

where $\alpha_j = \alpha_j^{(k)}$ are real numbers depending on k and $1 + 2 \sum_{j=1}^{k} |\alpha_j| \leq 2^k$.

Proof. For (a), known facts about partial fraction decompositions provide a decomposition of the given form for some (real) coefficients a_j and b_i, which sum to 1 by evaluation at $z = 0$. Positivity of a_j and b_i will follow from parts (b) and (c).

For (b), multiplying both sides of (7) by $(z-1)^m$, we get $(-1)^m(1 + z)^{-n}$, which is holomorphic except at $z = -1$. Comparing Taylor coefficients at $z = 1$ of orders $0, 1, ..., m - 1$ we get (8). For (c), a proof of (9) is symmetric.

Now for (d), clearly the left side has a partial fraction decomposition with constant term $(-1)^k$, letting $z \to +\infty$, and with coefficients times $(1-z)^{-j}$ and $(1+z)^{-j}$ for $j = 1, ..., k$. The coefficients of $(1-z)^{-j}$ and $(1+z)^{-j}$ are equal since interchanging z and $-z$ preserves the left side. Taking a binomial expansion of $z^{2k} = [(z^2 - 1) + 1]^k$, the fraction on the left side equals $\sum_{j=0}^{k} \binom{k}{j}(-1)^j(1-z^2)^{j-k}$. The sum of the absolute values of the coefficients in the binomial expansion is $(1+1)^k = 2^k$. Applying part (a) to each term $(1-z^2)^{j-k}$ for $j < k$ gives a sum of the stated form. For $j = k$ we get the constant term $(-1)^k$ as stated, and the bound 2^k holds, proving (d) and the lemma. $\square$

It turns out that the upper bound 2^k in part (d) can be improved to $2 \cdot (4/3)^k$. Such a bound and its sharpness will follow from Theorem 2 after the proof of Theorem 1.

Returning to the proof of Theorem 1, we have $A_v' = (-1)^v \alpha_v$ and $B_v' = (-1)^v \beta_v$ for each v, so $|A_v'| = |\alpha_v|$ and $|B_v'| = |\beta_v|$. Thus

$$|A_v| \leq |\alpha_v| \, |z_C|^{2k+v} \leq |\alpha_v| \cdot \delta^{-3k/2}$$

and, similarly, $|B_v| \leq |\beta_v| \cdot \delta^{-3k/2}$. Finally, we get that the coefficients in (6) are bounded by

(11)
$$\delta^{-k} + \sum_v (|A_v| + |B_v|) \leq 2^k/\delta^{3k/2}.$$

Clearly, $2^k/\delta^{3k/2} \leq 2^r/\delta^{3r/2}$ for $k = 1, ..., r$, which finishes the proof of (4).

So now, to prove (5), we can assume we have a sum $\sum_s a_s/(x - z_s)^{k_s} = 0$ for all real x where $k_s = 1, ..., r$, the pairs (k_s, z_s) are different for different s, $\Re z_s = 0$, $\delta \leq |\Im z_s| \leq 1/\delta$, and $\sum_s |a_s| < \infty$. The series converges uniformly in any compact subset of the complement of $\{z : \Re z = 0, \ \delta \leq |\Im z| \leq 1/\delta\}$.

By analytic continuation, g is holomorphic in the whole complement and 0 there. We will get a contradiction from that.

The following argument has been known apparently at least since the 1920's. Ross and Shapiro (2002), Proposition 3.2.2 p. 18, give such an argument for $r = 1$, but it extends directly to any r.

Let s be such that k_s is maximal, so we can assume $k_s = r$. It will be shown that $t^r f(t + iy_s) \to a_s$ as real $t \to 0$, by dominated convergence for sums. We have $|t^r a_v/(t + i(y_s - y_v))^{k_v})| \leq |a_v|$ for all v since $k_v \leq r$, so we have domination by a summable sequence. Moreover if $s \neq v$ the v term approaches 0 as $t \to 0$ either because $y_s \neq y_v$ or they are equal and $k_v < r$. For $v = s$ we get $t^r a_s/t^r \equiv a_s \to a_s$ as $t \to 0$. By this contradiction, the conclusion must hold for $d = 1$.

Now let $d \geq 2$. In this case there are still finitely many possibilities for the polynomials P_s. Call a set $A \subset \mathbb{R}^d$ *algebraic* if it is of the form $A = \{x : P(x) = 0\}$ for some polynomial P, not identically 0. It's easily seen that any countable union of algebraic sets in $\mathbb{R}^d$ has dense complement: since an algebraic set A is closed, it suffices to show that A is nowhere dense, then apply the Baire category theorem. If A were dense in a non-empty open set U it would include U. So $P \equiv 0$ on U and so P is identically 0, a contradiction.

Now consider the cube K_d of all vectors $\alpha = (\alpha_1, ..., \alpha_d)$ with $1 \leq \alpha_k \leq 2$ for all $k = 1, ..., d$. For each $\alpha \in K_d$, consider the function of one variable $f_\alpha(t) := f(t\alpha)$, $t \in \mathbb{R}$. For a dense set of values of $\alpha \in K_d$, for all s, the numerator of the s term in f_α is a non-zero multiple of t^{2k_s}, and the numbers $\alpha' C_s \alpha$ are distinct for different C_s, since these properties hold outside a countable union of algebraic sets. Choose and fix a value of α having these properties. Then we can apply the $d = 1$ case for that value of α, proving Theorem 1. $\square$

To improve the bound in Lemma 5(d), multiplying both sides of (10) by $(1 - z)^k$, we get

$$(12) \qquad \left[\frac{z^2}{z + 1}\right]^k = \sum_{v=0}^{k-1} \alpha_{k-v}(-1)^v(z - 1)^v + g(z)(z - 1)^k$$

for some function g holomorphic in a neighborhood of $z = 1$. It follows that $(-1)^j \alpha_{k-j}$ for $j = 0, 1, \ldots, k-1$ are the Taylor coefficients of order j of $[z^2/(z+1)]^k$ around $z = 1$ or equivalently, of $[(z+1)^2/(z+2)]^k = \left(z + \frac{1}{z+2}\right)^k$ around $z = 0$. Since the signs $(-1)^j$ don't affect the absolute values, we need to bound the sum of the absolute values of these Taylor coefficients through order $k - 1$. In Theorem 2, equation (16) we will see that $(-1)^j \alpha_j^{(k)} > 0$ for all $k = 1, 2, \ldots$ and $j = 0, 1, \ldots, k-1$.

Let c_i^n be the coefficients of the Taylor series of $\left(z + \frac{1}{2+z}\right)^n$ at $z = 0$, namely

$$\left(z + \frac{1}{z+2}\right)^n = c_0^n + c_1^n z + \cdots + c_n^n z^n + c_{n+1}^n z^{n+1} + \cdots.$$

Plugging in $z = 1$, we get that

(13)
$$\left(\frac{4}{3}\right)^n = c_0^n + c_1^n + \cdots + c_n^n + c_{n+1}^n + \cdots.$$

In what follows, $\binom{x}{k}$ is defined (as usual) as $x(x-1)\cdots(x-k+1)/k!$ for any real number x and integer $k \geq 0$, and as 0 otherwise (specifically, if $k < 0$).

Theorem 2. *For any integers $n \geq 0$ and $l \geq 0$*

(14)
$$c_{n+l+1}^n = \frac{(-1)^l}{2^{2n+l+1}} \sum_{a=0}^{l} \binom{l}{a}\binom{2n}{n-1-a}(-1)^a.$$

We also have

(15)
$$T_n := \sum_{l=0}^{\infty} c_{n+1+l}^n = \sum_{b=0}^{n-1} \frac{\binom{2n}{b}}{2^{2n}} \cdot \frac{3^b}{3^n} > 0.$$

Also for any $0 \leq l \leq n$

(16)
$$c_{n-l}^n = \frac{1}{2^{2n-l}} \sum_{a=l}^{n} \binom{a}{a-l}\binom{2n}{n+a} > 0.$$

In particular, for any integer $n \geq 0$

(17)
$$c_n^n = \frac{1}{2} + \frac{1}{2^{2n+1}}\binom{2n}{n}.$$

Let $S_n := \sum_{l=0}^{n-1} c_l^n = \sum_{l=0}^{n-1} |c_l^n|$. Then

(18)
$$\left(\frac{4}{3}\right)^n - \frac{1}{2} \geq S_n = \left(\frac{4}{3}\right)^n - \frac{1}{2} - \frac{1}{\sqrt{\pi n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

Proof. In this proof, identities (21) and (23) can be traced back to a classical identity given by Vandermonde (1772), $\sum_{k \in \mathbb{Z}} \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$, which is sometimes called Vandermonde's convolution. It holds for any values of r and s provided that m and n are integers. However, according to Graham, Knuth and Patashnik (1994, §5.1), this identity was known to Chu Shih-Chieh in China as early as 1303.

We will heavily use the following identity:

$$(19) \qquad 2^{-2n}\binom{2n}{n+a}(-1)^a = \sum_{b=0}^{n}\binom{n}{b}\binom{2b}{b+a}2^{-2b}(-1)^b,$$

for which we don't know a reference, so first let us show that it is valid. Note that we can rewrite it as follows:

$$2^{-2n}\binom{2n}{n+a}(-1)^{n+a} = \sum_{b=0}^{n}\binom{n}{b}(-1)^{n-b}\binom{2b}{b+a}2^{-2b}.$$

On the left is the coefficient of z^{n+a} in the polynomial $(z-1)^{2n}\,2^{-2n}$. Let us compute that coefficient in a different way:

$$(z-1)^{2n}\,2^{-2n} = \left(\frac{z^2-2z+1}{4}\right)^n = \left((-z)+\frac{(z+1)^2}{4}\right)^n.$$

Using the binomial theorem twice, we have that:

$$
\begin{aligned}
(z-1)^{2n}\,2^{-2n} &= \sum_{b=0}^{n}\binom{n}{b}(-z)^{n-b}\,2^{-2b}(z+1)^{2b}\\
&= \sum_{b=0}^{n}\binom{n}{b}(-1)^{n-b}\,2^{-2b}z^{n-b}\cdot\sum_{k=0}^{2b}\binom{2b}{k}z^k\\
&= \sum_{b=0}^{n}\sum_{k=0}^{2b}\binom{n}{b}\binom{2b}{k}(-1)^{n-b}\,2^{-2b}z^{n-b+k}.
\end{aligned}
$$

Now in order to get the coefficient of z^{n+a}, we need to plug in $k = b+a$, and this leads to (19).

Let us apply the binomial theorem to $\left(z + \frac{1}{z+2}\right)^n$. We obtain that

$$\left(z + \frac{1}{z+2}\right)^n = \sum_{b=0}^{n} \binom{n}{b} z^{n-b} (z+2)^{-b}$$

$$= \sum_{b=0}^{n} \binom{n}{b} z^{n-b} \sum_{k \geq 0} \binom{-b}{k} z^k 2^{-b-k}$$

$$= \sum_{b=0}^{n} \sum_{k \geq 0} \binom{n}{b} \binom{-b}{k} 2^{-b-k} z^{n-b+k}.$$

If $l = k - b - 1$, then $n - b + k = n + l + 1$, and we get a formula for the coefficient of z^{n+l+1}, namely

$$c_{n+l+1}^n = \sum_{b=0}^{n} \binom{n}{b} \binom{-b}{b+l+1} 2^{-2b-l-1},$$

or, using the fact that $\binom{-b}{b+l+1} = \binom{2b+l}{b+l+1}(-1)^{b+l+1}$, we get

(20) $$c_{n+l+1}^n = \sum_{b=0}^{n} \binom{n}{b} \binom{2b+l}{b+l+1} (-1)^{b+l+1} 2^{-2b-l-1}.$$

This formula holds for any value of l, but it is not convenient for our purposes. Let now l be non-negative. Equating the coefficients of z^{b+l+1} on both sides of $(z+1)^{2b+l} = (z+1)^l (z+1)^{2b}$, we obtain that

(21) $$\binom{2b+l}{b+l+1} = \sum_{a=0}^{l} \binom{l}{a} \binom{2b}{b+a+1}.$$

Now we rewrite (20) as follows:

$$c_{n+l+1}^n = \sum_{b=0}^{n} \binom{n}{b} (-1)^{b+l+1} 2^{-2b-l-1} \sum_{a=0}^{l} \binom{l}{a} \binom{2b}{b+a+1}$$

$$= (-1)^l \sum_{a=0}^{l} \binom{l}{a} 2^{-l-1} \left[\sum_{b=0}^{n} \binom{n}{b} \binom{2b}{b+a+1} 2^{-2b} (-1)^{b+1} \right]$$

$$= (-1)^l \sum_{a=0}^{l} \binom{l}{a} 2^{-l-1} \left[2^{-2n} \binom{2n}{n+a+1} (-1)^a \right]$$

using (19), which leads to (14).

Next, to prove (15), we have from (14):

$$\sum_{l=0}^{\infty} c_{n+1+l}^n = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2n+l+1}} \sum_{a=0}^{l} \binom{l}{a} \binom{2n}{n-1-a} (-1)^a$$

$$= \frac{1}{2^{2n+1}} \sum_{a\geq 0} \binom{2n}{n-1-a} (-1)^a \sum_{l\geq a} \binom{l}{a} (-2)^{-l}.$$

Note that if $x = -\frac{1}{2}$ then we have for any integer $a \geq 0$

$$\sum_{l\geq a} \binom{l}{a} (-2)^{-l} = \sum_{l\geq a} \binom{l}{a} x^l$$

$$= \frac{x^a}{a!} \sum_{l\geq a} l(l-1)\cdots(l-a+1) x^{l-a}$$

$$= \frac{x^a}{a!} \sum_{l\geq a} \frac{d^a}{dx^a} x^l = \frac{x^a}{a!} \frac{d^a}{dx^a} \sum_{l\geq 0} x^l$$

$$= \frac{x^a}{a!} \frac{d^a}{dx^a} \frac{1}{1-x} = \frac{x^a}{a!} \frac{a!}{(1-x)^{a+1}}$$

$$= (-1)^a \frac{2}{3^{a+1}}.$$

Here we added $1 + x + \cdots + x^{a-1}$, but since its ath derivative is zero, the equality still holds. Thus,

$$T_n = \sum_{l=0}^{\infty} c_{n+1+l}^n = \frac{1}{2^{2n}} \sum_{a\geq 0} \binom{2n}{n-1-a} 3^{-a-1}$$

(22)
$$= \sum_{b=0}^{n-1} \frac{\binom{2n}{b}}{2^{2n}} \cdot \frac{3^b}{3^n},$$

which is (15).

Next we want to prove (16). Let $l > 0$. Equating the coefficients of z^{b-l+1} on both sides of $(z+1)^{2b-l} = (z+1)^{-l}(z+1)^{2b}$, we obtain that

(23)
$$\binom{2b-l}{b-l+1} = \sum_{a} \binom{-l}{a-l} \binom{2b}{b-a+1}.$$

The sum in the last display is finite since $\binom{-l}{a-l}$ equals 0 for $a < l$ and $\binom{2b}{b-a+1}$ is 0 for $a > b+1$. Then equation (20) can be transformed as

follows:

$$c^n_{n-l+1} = \sum_{b=0}^{n} \binom{n}{b}(-1)^{b-l+1}2^{-2b+l-1} \cdot \sum_{a} \binom{-l}{a-l}\binom{2b}{b-a+1}$$

$$= (-1)^l \sum_{a} \binom{-l}{a-l}2^{l-1}\left[\sum_{b=0}^{n}\binom{n}{b}\binom{2b}{b+a-1}2^{-2b}(-1)^{b+1}\right]$$

$$= (-1)^l \sum_{a} \binom{-l}{a-l}2^{l-1}\left[\binom{2n}{n+a-1}2^{-2n}(-1)^a\right]$$

by (19). Replacing $\binom{-l}{a-l}$ by $\binom{a-1}{a-l}(-1)^{a-l}$, we obtain:

$$c^n_{n-l+1} = \sum_{a} \binom{a-1}{a-l}\binom{2n}{n+a-1}2^{-2n+l-1},$$

which leads to (16).

To prove (17), from (16) we have $c^n_n = 2^{-2n}\sum_{a=0}^{n}\binom{2n}{n+a}$, so

$$2^{2n}c^n_n = \frac{1}{2}\sum_{a=0}^{n}\left(\binom{2n}{n+a}+\binom{2n}{n-a}\right)$$

$$= \frac{1}{2}\left(\binom{2n}{n}+\sum_{a=0}^{2n}\binom{2n}{a}\right) = \frac{2^{2n}+\binom{2n}{n}}{2},$$

which leads to (17).

Recall that $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$. To study the behavior of c^n_i for n large, we have the following, which follows directly from Stirling's formula:

$$(24) \qquad \frac{1}{2^{2n}}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

This formula is a special case of the (local) central limit theorem for binomial probabilities first found by de Moivre (1733). Formula (24) immediately implies that

$$(25) \qquad c^n_n - \frac{1}{2} \sim \frac{1}{2\sqrt{\pi n}}.$$

Now in (15), $T_n = \frac{1}{3 \cdot 4^n}\binom{2n}{n-1}R_n$ where, bounding by a geometric series,

$$R_n := 1 + \frac{n-1}{3(n+2)} + \frac{(n-1)(n-2)}{3^2(n+2)(n+3)} + \cdots \leq \frac{3}{2}$$

and $R_n \to 3/2$ as $n \to \infty$. It follows by Stirling's formula that $T_n \sim 1/(2\sqrt{\pi n})$ as $n \to \infty$.

We have $c_l^n > 0$ for $l = 0, 1, \ldots, n-1$ by (16). The bound on the left side of (18) follows from (13), (15) and (17). The asymptotic form on the right follows by using in addition (25) and $T_n \sim 1/(2\sqrt{\pi n})$. This finishes the proof of the theorem. $\qquad\square$

Remark. In the proof of Theorem 1 for $d = 1$, one argument was mentioned to be known according to Ross and Shapiro (2002, p. 18). They say that the argument is closely related to an example of Poincaré (1883). Ross and Shapiro don't give details of the argument (although it is short, as given in our proof), they just say the result "can be obtained from the dominated convergence theorem." They also are treating points converging to the unit circle in the plane along radii through 0, but the case of a line instead of the circle if anything seems easier.

Another interesting fact mentioned by Ross and Shapiro (2002) shows that, again for $d = 1$, for such a proof, of uniqueness of z_s, k_s and a_s to work, the condition that all the z_s are on a line segment or smooth curve is really used. In fact, there is a bounded sequence $\{z_j\}$ of distinct complex numbers with $|z_j| > 1$ for all j, such that the set of all limits of subsequences of $\{z_j\}$ is exactly the unit circle $\{z : |z| = 1\}$, and there exist coefficients $a_j \in \mathbb{C}$ with $\sum_j |a_j| < \infty$, for which the series $f(z) := \sum_j a_j/(z - z_j)$ converges to 0 for all z with $|z| < 1$, whereas on $|z| > 1$, f is a meromorphic function with a pole of order 1 at each z_j. Such an example is given in Ross and Shapiro (2002, Theorem 4.2.5), where in that theorem we take the special case that G is the open unit disk (just as in the proof given by Ross and Shapiro) and $f \equiv 0$. Such examples are said to have been first found by Wolff (1921).

As a corollary of Theorem 1, in Lemma 3 in the special case $j = 1$ we get $\|f\|_{\delta,r}^{*,1} = 1$, showing that part (a) is sharp and improving on part (c) in that case.

Remark. The equality is not true for general j, however. Let $0 < \delta < \alpha < 1$ and let
$$f(x) = x^4/[(1 + \alpha x^2)(1 + \alpha^{-1} x^2)]$$
so that $f \in \mathcal{F}_{\delta,2}^{(2)}$. Then
$$f(x) \equiv \frac{\alpha}{1 - \alpha^2} \left(\frac{x^2}{1 + \alpha x^2} - \frac{x^2}{1 + \alpha^{-1} x^2} \right),$$
so $\|f\|_{\delta,2}^{*,2} \le 2\alpha/(1 - \alpha^2)$. This can be less than 1, in fact it's arbitrarily small for small δ and α.

In view of Theorem 1, one might expect that if

$$\sum_s a_s/[(1+C_s x^2)^{u_s}(1+D_s x^2)^{v_s}] = 0$$

with $\{C_s, D_s\} \neq \{C_t, D_t\}$ for $s \neq t$ then all $a_s = 0$. Unfortunately, this is wrong, as shown by the following simple example:

$$\frac{2}{(1+x^2)(1+3x^2)} - \frac{1}{(1+x^2)(1+2x^2)} = \frac{1}{(1+2x^2)(1+3x^2)}.$$

For any $r = 1, 2, ...,$ $d = 1, 2, \ldots$, any $P \in \mathcal{MM}_{2r}$, and any $C \neq D$ in $\mathcal{W}_\delta$, let

$$f_{P,C,D}(x) := f_{P,C,D,r}(x) := f_{P,C,D,r,d}(x) :=$$

$$\frac{P(x)}{(1+x'Cx)^r} - \frac{P(x)}{(1+x'Dx)^r}.$$

By Lemma 2, for C fixed and $D \to C$ we have $\|f_{P,C,D,r}\|_{\delta,r+1}^{*,2} \to 0$. The following shows this is not true if $r+1$ in the norm is replaced by r, even if the number of different C_s's in the denominator is allowed to be as large as possible, namely r:

Proposition 2. *For any $r = 1, 2, ...,$ $d = 1, 2, \ldots$, $P \in \mathcal{MM}_{2r}$, and $C \neq D$ in $\mathcal{W}_\delta$, we have*

$$\|f_{P,C,D,r}\|_{\delta,r}^{*,r} = 2.$$

Proof. First let $d = 1$. Suppose that

$$(26) \quad f_{P,C,D,r}(x) \equiv f(x) := x^{2r} \sum_{s=1}^{\infty} \frac{a_s}{(1+x'C_{s1}x)\cdots(1+x'C_{sr}x)},$$

where all $C_{sk} \in (\delta, 1/\delta)$, and $\sum_s |a_s| < \infty$. Then the series summing to f in (26) will converge uniformly in all real x and moreover for x replaced by complex z with $|\Im z| \leq \gamma$ for any γ with $0 < \gamma < \sqrt{\delta}$. Let $J := \{z \in \mathbb{C} : \Re z = 0, \ \delta \leq |\Im z| \leq 1/\delta\}$. As seen previously, the series also will converge uniformly on any compact subset of the complement of J. A point $z_0 = iy$ in J (so that $\delta \leq |y| \leq 1/\delta$ and y is real) will be called a *semipole of order r* of a complex-valued function g defined on $U := \mathbb{C} \setminus J$, where r is a positive integer, if for some $c \in \mathbb{C}$ with $c \neq 0$,

$$(27) \quad \lim_{t \downarrow 0} t^r g(iy + t) = c.$$

Let

$$F(z) := z^{2r}\left[\frac{1}{(1+Cz^2)^r} - \frac{1}{(1+Dz^2)^r} - \sum_{s=1}^{\infty} \frac{a_s}{\prod_{k=1}^{r}(1+C_{sk}z^2)}\right],$$

convergent to 0 on the complement of J and uniformly on compact subsets of the complement. Let $z := x + i/\sqrt{C}$ and $x \to 0$, and consider the behavior of $x^r F(z)$. For any $B > \delta$ such as $B = C_{sk}$, C or D, we will have $1 + Bz^2 = 1 - B(C^{-1} - x^2) + 2iBx/\sqrt{C}$, which has absolute value at least $2\delta|x|/\sqrt{C}$. Thus $|x/(1 + Bz^2)| \le \delta^{-3/2}$ and for the limit we have dominated convergence for sums. Moreover, the limit is 0 except for terms with $C_{sk} = C$ for all $k = 1, ..., r$. Combining the a_s for such terms into one, we must have $a_s = 1$ for such an s. We can do likewise for D in place of C, which finishes the proof for $d = 1$.

For $d \ge 2$, let S^{d-1} denote the unit sphere centered at the origin in $\mathbb{R}^d$. We will use the Haar measure (rotationally invariant Borel probability measure) μ on S^{d-1}. For any $e \in S^{d-1}$, define a function g_e of a real variable t by

$$g_e(t) := f(te) = t^{2r} \sum_{s=1}^{\infty} \frac{a_s P_s(e)}{(1 + t^2(e'C_{s1}e)) \cdots (1 + t^2(e'C_{sr}e))}.$$

For any fixed matrix M, e.g. $M = C_{sk}$, let $A_{M,C}$ be the set of all e in S^{d-1} for which $e'Ce = e'Me$, or $e'(C - M)e = 0$. If $C \ne M$, then not all eigenvalues of $C - M$ are zero, which entails that the codimension of $A_{M,C}$ is at least 1. Thus, we obtain that $\mu(A_{M,C}) = 0$ provided that $M \ne C$, and likewise $\mu(A_{M,D}) = 0$ for $M \ne D$.

For each e such that $P(e) \ne 0$ and $e'(C - D)e \ne 0$, as are true for almost all e, the function $f_{e,P,C,D}(t) := f_{P,C,D}(te)$ has two poles of order r at the points $z_e^{\pm} = \frac{\pm i}{\sqrt{e'Ce}}$. Then, g_e must have semipoles of order r at the same two points, as shown in the $d = 1$ case. The same is true for the points $\zeta_e^{\pm} = \frac{\pm i}{\sqrt{e'De}}$.

For μ-almost all e, we have $e'C_{sj}e \ne e'Ce$ for all s and j such that $C_{sj} \ne C$, and $P(e) \ne 0$. For any such e, contributions to the limit c at $iy = z_e^{\pm}$, via dominated convergence for sums as in the proof of Theorem 1, can come only from terms in (26) in which $C_{sj} = C$ for all $j = 1, ..., r$. Let $MM(r) := MM(r, d)$ be the finite number of different monomials in $\mathcal{MM}_{r,d}$, namely $MM(r, d) = \binom{2r+d-1}{d-1}$, and call them $Q_1, Q_2, \ldots, Q_{MM(r)}$. Let $b_j = a_s$ when $P_s = Q_j$ and $C_{si} = C$ for all $i = 1, ..., r$. Some of these coefficients may be 0. We then get an equation

$$P(e) = \sum_{j=1}^{MM(r)} b_j Q_j(e)$$

holding for μ-almost all e, by applying (27) to both sides of (26) and using dominated convergence for sums, as in the proof of Theorem 1, or the proof for $d = 1$. Multiplying both sides by t^{2r} we get a polynomial

equality holding almost everywhere on $\mathbb{R}^d$ and so everywhere. Since different monomials are linearly independent, it follows that $b_j = 1$ for the j such that $P = Q_j$ and other $b_j = 0$. The same argument with D in place of C gives that $a_s = -1$ for another coefficient in (26) and so shows that $\|f_{P,C,D,r}\|_{\delta,r}^{*,r} \geq 2$. From the original expression of $f_{P,C,D,r}$ it's clear that the norm is ≤ 2, so it follows that it equals 2.

Here, in (26) we could also consider on the right terms where the numerators and denominators were of the same form but of total degree $2r - 2, 2r - 4, ..., 0$ rather than $2r$. There would be no essential change in the proof since such terms do not affect or contribute semipoles of order r. The proposition is proved. $\qquad\square$

Remark. In Lemma 1 we had two-sided bounds in the sup norm for functions in $\mathcal{G}_{\delta,r}$ and in Lemma 3(b), the sup norm is bounded above in terms of the $\|\cdot\|_{\delta,r}^{*,j}$ norm. There is no such bound in the reverse direction and in fact, for example, $Y_{\delta,r}^1$ is not complete in the sup norm in dimension $d = 1$. To see this, for a fixed $\delta \in (0,1)$ let $a := \delta$ and $b := 1/\delta$. For $n = 1, 2, ...,$ let $C_{jn} := a + (j(b-a)/n) - 1/(2n)$ for $j = 1, ..., n$. Then $a < C_{jn} < b$ for each j. Let $f_n(x) := \sum_{j=1}^n x^{2r}/[n\{1 + (C_{jn}x^2)^r\}]$. Then it is not hard to show that f_n converges in the sup norm as $n \to \infty$ to the function $g(x) := x^{2r} \int_a^b dC/[(b-a)(1 + Cx^2)^r]$, but g is not in $Y_{\delta,r}^1$, because g has no semipoles.

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